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ABSTRACT

We investigate a random proposer bargaining game with a dead line. A bounded time interval is divided into bargaining periods of equal length and we study the limit of the subgame perfect equilibrium outcome as the number of bargaining periods goes to infinity while the dead line is kept fixed. This limit is close to the Raiffa solution when the time horizon is very short. If the dead line goes to infinity the limit outcome converges to the time preference Nash solution. The limit outcome is given an axiomatic characterization as well.

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1 Introduction

We investigate a random proposer bargaining game with a dead line. A bounded time interval is divided into bargaining periods of equal length and we study the limit of the subgame perfect equilibrium outcome as the number of bargaining periods goes to infinity while the dead line is kept fixed. This limit is close to the Raiffa solution when the time horizon is very short. If the dead line goes to infinity the limit outcome converges to the time preference Nash solution (Chae 1993). The limit outcome is given an axiomatic characterization as well.

There are several papers analyzing the problem how the outside alternatives available to the bargainers affect the outcome of the bargaining game (see *e.g.* Shaked and Sutton (1984) and Binmore, Rubinstein, and Wolinsky (1986)). These studies show that if impasse is the best alternative, there is no chance of bargaining break down and the bargaining possibility remains indefinitely, then the best alternative has no effect. A natural setting where the best alternative outcome could affect the bargained outcome is the case when the surplus associated with the bargaining problem vanishes in finite time and after that bargainers have to take their best alternatives.

It should be noted that we are not dealing with the issues concerning the "deadline effect" (*c.f.* Fershtman and Seidman 1993; Roth, Murnighan and Schoumaker 1988; Ma and Manove 1993; Simsek and Yildiz 2008). In this paper finite horizon and dead line mean the same thing: the bargaining opportunities vanish in finite time. An instance for such dead line is given by a special tax treatment offered up till a certain date. If this is the case of a tax-break for a corporate merger, then after a certain date, such a break becomes not applicable for firms bargaining over the terms of the merger.

It is well known that as the deadline tends to infinity, the subgame perfect equilibrium outcome of the random offer or alternating offer bargaining game converges to the one of the corresponding infinite horizon game (Binmore 1985). Moreover this outcome converges to the Nash bargaining solution as the discount factor goes to one. (All these results are established in Rubinstein (1982) and Binmore (1987)).

Following Stahl (1972), Sjostrom (1991) analyzed a related model. A finite time interval is divided into bargaining periods of equal length. When the number of periods goes to infinity the limit of bargaining outcomes remains close to the (discrete) Raiffa solution (Raiffa 1953, 2002). A simplified version of this solution for a piecewise linear problems was discussed in Raiffa (1953). An axiomatic foundation for this solution is given in Salonen (1986). The Raiffa solution can be obtained as the limit of the following procedure. In the beginning, each agent demands his "ideal point" which is the best outcome for him satisfying individual rationality constraints. Then the average of these two demands are given as a reference point, and players make new demands with the constraint that nobody gets less than this reference point, and so on. The limit outcome of this procedure is the Raiffa solution[1].

Sjostrom (1991) proved that without discounting, the subgame perfect equilibrium outcomes converge to the Raiffa solution as the number of bargaining periods increases without limit. He showed also that with very little discounting the limit outcome lies very close to the Raiffa solution. Of course with constant non-negligible discounting this result does not restrain the outcome much and in fact as the dead line goes to infinity, the outcome converges to the Nash solution. In this paper, we try to reconcile these two observations by investigating more closely the limiting outcomes for finite horizon and positive discounting.

In the latter half of the paper, we attempt to axiomatize the obtained limit solution, in part due to the diversity of the solutions arising from the strategic approach. To our knowledge, not much effort has been devoted to include the time dimension in the axiomatic bargaining theory.

The paper is organized in the following way. Notation and some preliminary observations are presented in Section 2. The main results concerning the strategic model are given in Section 3. Section 4 is devoted to the axiomatic approach and Section 5 concludes.

2 Preliminaries

A 0-1 normalized *bargaining problem* is given by a compact and convex set $S \subset \mathbb{R}^2_+$ that contains the origin. The Pareto frontier of S, PS, is given by $y_2 = F(y_1)$ such that $1 = F(0),\ 0 = F(1)$, where F is strictly concave, continuous, and decreasing. Further, we assume that F and F^{-1} are defined and continuously differentiable on some open $U \supset [0,1]$. For a splitting-a-dollar problem, F is generated by concave, continuous, strictly increasing and continuously differentiable functions u_1 and u_2 on [0,1] with $u_i(0) = 0$ and $u_i(1) = 1$ $(i = 1, 2)$, such that given y_1, with $y_1 = u_1(q)$ for some $q \in [0, 1]$, $F(y_1) = u_2(1 - q)$. The Nash solution here is given by

$$\arg \max_{(y_1,y_2)\in S} y_1 y_2 = \{N^*(S)\}.$$

Given a deadline $T > 0$ and a positive integer m, let $T/m = \Delta_m$ which is the length of a period. We count periods backwardly so that m is the first round of the bargaining game. The rules of the sequential bargaining game with a random proposer are as follows. At each period t Nature chooses a proposer with equal probabilities. The chosen proposer makes a proposal q_t in $[0, 1]$, and another player replies by "Yes" or "No". If the reply is "Yes", then the game ends with an outcome described by (t, q_t). If the reply is "No", then the

[1] Continuous counter part of this procedure is also proposed by Raiffa and discussed in Peters and Van Damme(1991) (see also Livne(1989)).

game moves into period $t-1$ if $t>0$, or ends with payoffs (a_1, a_2) (evaluated at $t=0$) if t $=0$. We assume that $a=(a_1,a_2)\in S$ but unlike Sjostrom (1991), $a=(0,0)$ is not required.

Players maximize discounted expected utilities. The discount rate $r>0$ is common to players with the discount factor $\delta=\delta_{r,m}=e^{-r\Delta_m}$, which we often call "$\delta$ given r and m".

Next, we define the Raiffa solution $R(S,a)$ with respect to termination payoffs $a=(a_1,a_2)$. Let $z^0=(a_1,a_2)$. Given $z^{j+1}=\left(\frac{z_1^j+F^{-1}(z_2^j)}{2},\frac{z_2^j+F(z_1^j)}{2}\right)$, $R(S,a)=\lim_{j\to\infty}(z^j)$ and as in Sjostrom (1991), also one can write

$$R_1(S,a)=\frac{1}{2}\sum_{j=0}^{\infty}\left(F(z_2^j)-z_1^j\right)+a_1$$

and similarly for agent 2.

We investigate the subgame perfect equilibrium outcome of this game. To define this concept, we have to define strategies first. The game starts at period m and players get their outside alternatives at the end of period 0 if no agreement is reached prior to that.

We define first histories. Let $\phi^m=\emptyset$, and for $n=0,\ldots,m-1$ let $\phi^n=(i^t,w^t,\rho^t)_{t=n+1}^m$, where in round t player i^t is the proposer, w^t is the offer made, and $\rho^t=$ "No" is the reply. So ϕ^n is the *history* in the beginning of period n before the proposer has been selected. Then a period n history for the proposer i^n is (ϕ^n,i^n) and a period n history for the responder is (ϕ^n,i^n,w^n).

A *strategy* of player $i\in\{1,2\}$ is a mapping σ^i that maps each period n history for proposer i into $[0,1]$ and each period n history for responder i into {"Yes","No"}. A strategy profile (σ_1,σ_2) is a *subgame perfect equilibrium* if conditional on each history, σ_i is optimal given $\sigma_j (j\neq i)$. Let us characterize now the equilibrium outcome.

If agent 1 is the proposer at the 0-th period, then he makes an offer $F^{-1}(a_2)$. The corresponding allocation $(F^{-1}(a_2),a_2)$ is accepted by agent 2. If agent 1 is the responder, then 2 makes an offer $F(a_1)$ which is accepted. This yields the expected payoffs, or the *continuation values* for period 0:

$$(z_1^0,z_2^0)=\left(\frac{1}{2}\left(a_1+F^{-1}(a_2)\right),\frac{1}{2}\left(F(a_1)+a_2\right)\right).$$

For $n\geq 1$ we have

$$(z_1^n,z_2^n)=\left(\frac{1}{2}\left(\delta z_1^{n-1}+F^{-1}\left(\delta z_2^{n-1}\right)\right),\frac{1}{2}\left(F\left(\delta z_1^{n-1}\right)+\delta z_2^{n-1}\right)\right). \tag{1}$$

As usual, an immediate agreement obtains and so the subgame perfect equilibrium payoffs are then (z_1^m,z_2^m), and we would like to investigate their values in the limit as Δ_m tends to 0. We write $z(t)$ for $\lim\{z^{n(m)}:\Delta_m\to 0, n(m)/m\to t, t>0\}$, provided that the limit exists.

Example 1. Suppose that the frontier PS is given by $y_1+y_2=1$. Writing $z^n=(z_1^n,z_2^n)$, (1) becomes

$$z^{n+1}=\left(\frac{1}{2}\left(\delta z_1^n+1-\delta z_2^n\right),\frac{1}{2}\left(1-\delta z_1^n+\delta z_2^n\right)\right)$$

and if z^n lies on the frontier, (which is the case for $n>0$)

$$z^{n+1}-z^n=(1-\delta)\left(\frac{1}{2}-z_1^n,\frac{1}{2}-z_2^n\right).$$

Directly solving, one obtains for $i=1,2$ that

$$\begin{aligned}z_i^{n+1}&=z_i^n+(1-\delta)\left(\frac{1}{2}-z_i^n\right)\\&=\frac{1-\delta}{2}+\delta z_i^n.\end{aligned}$$

Solving this recursive equation gives us

$$z_i^{n+1}=\delta_i^{n+1}z_i^0+\frac{1-\delta^{n+1}}{2}.$$

Recall that $\delta^n=\left(e^{-rT/m}\right)^n$, where $T/m=\Delta_m$ is the length of the time interval between two consecutive offers. In the limit, as this time interval Δ_m goes to 0, we have the solution

$$z=e^{-rT}R(S,a)+\left(1-e^{-rT}\right)N^*(S).$$

Note that $N^*(S)=(1/2,1/2)$ and $R(S,a)=z^0$ in this case.

One could replace the difference equation by a differential formula, i.e.

$$Dz^n=(1-\delta)\left(\frac{1}{2}-z_i^n\right) \tag{2}$$

or

$$\frac{d\log\left(\frac{1}{2}-z_i^t\right)}{dt}=r$$

in the limit. We chose this representation of showing all allocations in each period by respective current values, because this conforms with the representation given in terms of differential equation by Coles and Wright (1998) and also it provides a nice interpretation that the adjustment is made toward the "global" Nash solution $N^*(S)$, at a rate proportional to the difference between the current value and Nash solution. However the present value representation gives also a nice picture. In this example one sees that the process jumps from $e^{-rT}a$ to $e^{-rT}R(S,a)$ and from there on it proceeds linearly to z with a direction of $(1/2,1/2)$.

For a general linear frontier $y_1/a + y_2/b = 1$, we shall have

$$z^n = \left(\delta z_1^{n-1} + \frac{a(1-\delta)}{2}, \delta z_2^{n-1} + \frac{b(1-\delta)}{2}\right)$$

or as above

$$z^n - z^{n-1} = \left((1-\delta)\left(\frac{a}{2} - z_1^{n-1}\right), (1-\delta)\left(\frac{b}{2} - z_2^{n-1}\right)\right)$$

and

$$Dz^n = (1-\delta)(N^*(S) - z^n) \tag{3}$$

where N^* is $(a/2, b/2)$.

3 The limit solution

The way differential equations were used in Example 1 turns out to be very convenient in solving the general case as well. In fact Coles and Wright (1998) utilized this approach in their analysis general random proposer games under non-stationary environments (see also Coles and Muthoo 2003; McLennan 1988; Binmore 1987). We apply their result to the case with a jump in the agreement set at the deadline. For a problem with a smooth Pareto frontier, denote by f the derivative $dF(y_1)/dy_1$, $y = (y_1, y_2) \in PS$,

Here, the relevant "local" Nash solution becomes a function of y, given by

$$(N_1(y), N_2(y)) = \left(\frac{y_1 + \frac{y_2}{-f}}{2}, \frac{-fy_1 + y_2}{2}\right)$$

and hence (2) would become

$$Dz^n = (1-\delta)(N(z^n) - z^n) \tag{4}$$

Thus, in general, the adjustment is made toward the local Nash solution at a rate proportional to the difference, and along the way, the local Nash solution itself moves toward the global Nash solution. We call the solution of (4) as the *limit solution* or the *limit outcome.*

Our main result is

Theorem 1 *Given $S, a, T,$and r the limit outcome is $x = (x_1, F(x_1))$ such that*

$$T = \int_{R_1(S,a)}^{x_1} \frac{1}{r(N_1(z_1, F(z_1)) - z_1)} dt$$

when $R(S,a) \neq N^*(S)$ and if $R(S,a) = N^*(S), x = N^*(S)$.

Note that if $R(S,a) = N^*(S)$,then $Dz^n = 0$ and we obtain the theorem immediately. In the case where $R(S,a) \neq N^*(S)$, showing that $z_1(T) = \int_0^T r(N_1(z(t)) - z_1(t))\, dt + R_1(S,a)$ holds is equivalent to prove the theorem. We shall prove this claim in several steps. We denote by $\|\cdot\|$ the absolute sum norm: $\|x\| = \sum_i |x_i|$.

Fix $x \in S$ and the Raiffa solution $R(S,a)$.

Lemma 1 *Given $\varepsilon > 0$, there is $\overline{\Delta} > 0$ and n with respect to a such that for $\Delta < \overline{\Delta}$,*

$$\|R(S,a) - z^n\| < \varepsilon$$

holds.

Proof. We first give names to the mappings defining vectors z^n:

$$G_0(z) = \left(\frac{1}{2}\left(z_1 + F^{-1}(z_2)\right), \frac{1}{2}\left(z_2 + F(z_1)\right)\right),$$
$$G(z) = G_0(\delta z).$$

Define also

$$G_0^\nu = \underbrace{G_0 \circ \cdots \circ G_0}_{\nu} \text{ and } G^\nu = \underbrace{G \circ \cdots \circ G}_{\nu}.$$

Since G_0 and G are continuous, so are G_0^ν , and G^ν. As δ tends to 1, G tends to G_0. Therefore $G_0^\nu(z)$ tends to $G^\nu(z)$ given ν and z. Thus given $\varepsilon > 0$, there is Δ_m and n so that $\|G_0^\nu(z) - R(S,a)\| < \varepsilon/2$ and $\|G_0^{n-1}(z) - G_0^{n-1}(z)\| < \varepsilon/2$. Since $G^{n-1}(z) = z^n$, $\|R(S,a) - z^n\| < \varepsilon$, as desired. ■

Next, we define z to be "ε -close to PS", if there is y in PS, the Pareto set of S, such that $\|y - z\| < \varepsilon$.

Lemma 2 *Given $\varepsilon > 0$, there is $\overline{\Delta}$ so that if $\Delta < \overline{\Delta}$, then it holds that if z^n is ε -close to PS, then δz^{n+1} is ε -close to PS.*

Proof. Write z for $G_0(z)$ and z' for z^{n+1}. Also write α for $F(z_1') - z_1'$ and β for $F^{-1}(z_2') - z_2'$.

Noting that z' is $\varepsilon/2$ -close to the frontier if z is ε -close to the frontier.

We have

$$\alpha + (1-\delta) z_1' + \frac{\alpha}{\beta}(1-\delta) z_2' + \beta + (1-\delta) z_2' + \frac{\beta}{\alpha}(1-\delta) z_1'$$
$$\leq \alpha + \beta + (1-\delta)\left(2 + \frac{\beta}{\alpha} + \frac{\alpha}{\beta}\right).$$

(If $\alpha = 0$ or $\beta = 0$, then α/β or β/α should be substitutet by the values of gradients of F or F^{-1} at z.) Since F is C^1, one can find a bound on $\beta/\alpha + \alpha/\beta$, say B. Choosing appropriate $\overline{\Delta}$, one can assure that $(1-\delta)(2+B) < \varepsilon/2$ for each $\Delta < \overline{\Delta}$.

For z in $S \backslash PS$, define

$$h(z) = \frac{F(\delta z_1) - \delta z_2}{F^{-1}(\delta z_2) - \delta z_1}.$$

h is continuous and as z gets close to $z' \in PS$ and $\Delta \to 0$, $h(z)$ converges to $F'(z')$, since F is C^1. Given $\varepsilon' > 0$ (and ε) for sufficiently small Δ, we have $|h(z) - F'(z'')| < \varepsilon'$ for each z and z'' with $z'' \in PS$ and $\|z - z''\| < \varepsilon$. ■

Proof of the Theorem. Now, since Coles and Wright (1998)'s result applies as the process yields the two outcomes $(\delta z_1^n, F(\delta z_1^n))$ and $(F^{-1}(\delta z_2^n), \delta z_2^n)$, satisfying the backward induction formula are ε-close to each other for sufficiently large n, their result implies the formula in the theorem. ■

The solution obtained is located between the Nash and the Raiffa solution. A larger T and a lower r moves the solution toward the Nash solution and a higher a_i given a_j $(i \neq j)$, shifts the Raiffa solution to the advantage of i and hence the solution changes in the same direction.

Example 2. For some cases, one can compute the solution explicitly. Let $u_1(w) = w^\alpha$ and $u_2(w) = w^\beta$ with $0 < \alpha, \beta \leq 1$ where w is the amount of money in the "divide-a-dollar" problem (these functions are not C^1 at the boundary, and so a is restricted to the interior of S). Then

$$F(u) = (1 - u^{1/\alpha})^\beta \text{ and } \frac{-1}{u + F/F'} = \frac{u^{1/(\alpha-1)}}{\alpha/\beta - (1+\alpha/\beta)u^{1/(\alpha-1)}}$$

so that

$$T(z_1) = -\frac{2}{r(1+\alpha/\beta)} \log \left[\frac{\alpha/\beta - (1+\alpha/\beta)z_1^{1/\alpha}}{\alpha/\beta - (1+\alpha/\beta)(R_1^a)^{1/\alpha}} \right]$$

or

$$x_1 = \frac{\left\{ \left[\alpha/\beta - (1+\alpha/\beta)(R_1^a)^{1/\alpha} \right] e^{-(1+\alpha/\beta)rT/2} \right\}^\alpha}{(1+\alpha/\beta)^\alpha}.$$

A change in the bargaining protocol affects the solution, especially through the change in the limit solution as T vanishes. A sequential bargaining game with player i having the last say yields the solution given by the same formula but the lower limit of the integral replaced by the i -th coordinate of the i's dictatorial solution instead of that of the Raiffa solution. Under the random proposer protocol with unequal probabilities, the solution is modified according to these probabilities. The Raiffa solution is changed so that the speed of adjustment in the integral is modified, and the local and global Nash solutions are replaced by the asymmetric Nash solutions with the weights given by these probabilities.

4 The axiomatization

One could view the limit solution as a function of a or $e^{-rT}a$. This solution satisfies sensitivity properties with respect to the "threat point" as well as the individual rationality property with respect to $e^{-rT}a$. Apparently, IIA property is not met because $R(S,a)$ does not satisfy it. In the rest of this section, we show an attempt to axiomatize this solution based on the axiomatization of the discrete Raiffa solution by Salonen (1986).

The class of problems we consider is $\Psi = \{B = (S, a, T, r) \mid S \subset \mathbb{R}^2_+, a \in S, T > 0, r > 0\}$, where S is a compact and convex subsets such that weakly Pareto optimal elements are Pareto optimal. Thus the class is slightly larger than the one we worked on earlier, where in order to utilize Coles and Wright (1998)'s result, we assumed smooth Pareto frontiers. We extend the definition of the "derivative" f of the Pareto boundary function F by first defining $f(z_1) = F'(z_1)$ if F is differentiable at z_1. For other values of z_1 we proceed as follows. Denote by D the set of all points z_1 at which F is differentiable. Then let $f(z_1) = \lim\{f(z_1') \mid z_1' \uparrow z_1, z_1' \in D\}$ if $z_1 < N_1^*(S)$; $f(z_1) = \lim\{f(z_1') \mid z_1' \downarrow z_1, z_1' \in D\}$ if $z_1 > N_1^*(S)$; and $f(z_1) = y$ such that $N(z_1, F(z_1)) = N^*(S)$ if $z_1 = N_1^*(S)$ (recall that $N(z_1, F(z_1))$ is the *local* Nash solution at $(z_1, F(z_1))$.

For a later use, we denote by Γ the set of pairs (S, a) such that $B = (S, a, T, r) \in \Psi$, for any $r > 0$, and $T > 0$. Extension of our solution to this class is of no problem because the function representing the Pareto frontier of S is differentiable $a.e.$, for all $(S, a, T, r) \in \Psi$, and so the integral defining the solution is still well defined.

A solution φ to bargaining problems in Ψ maps each $B = (S, a, T, r) \in \Psi$ to an element of S. We use the Hausdorff metric on compact subsets of $\mathbb{R}^2_+$ for emasuring the distance between bargaining problems. A positive affine transformation (α, β) on $\mathbb{R}^2$ with $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2_{++}$ and $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$ is defined by $(\alpha, \beta)(x) = (\alpha_1 x_1 + \beta_1, \alpha_2 x_2 + \beta_2)$ for any $x \in \mathbb{R}^2$. Denote by $(\alpha, \beta)B = ((\alpha, \beta)S, (\alpha, \beta)a, T, r)$ the resulting problem when a positive affine transformation (α, β) is applied to a problem $B = (S, a, T, r)$. Consider the following three basic properties for solutions on Ψ.

Efficiency (E). $\varphi(B) \in PS$, for all $B \in \Psi$.

Continuity $(Cont)$. φ is continuous with respect to a, T, r, and S, for all $B = (S, a, T, r) \in \Psi$.

Scale Invariance (SI). $\varphi((\alpha, 0)B) = (\alpha, 0)\varphi(B)$ for each problem $B \in \Psi$, and for any positive affine transformation $(\alpha, 0)$.

Next we introduce a key decomposition property, called the *end phase evaluation property*, that we utilize to single out a solution. Formally, we call the vector

$$g(S, a) = \lim_{T \to 0} \varphi(S, a, T, r) \in S$$

an *end phase evaluation* given $(S, a) \in \Psi$, if the limit on the right hand side exists and is independent of r.

By this concept, we try to pin down the bargaining outcome when there is an infinitesimally short period of time within which bargainers can exchange offers and counter offers. This outcome could be different from the one with an immediate dead line, *i.e.*, when there is time for a single offer only. It is quite natural to assume that this outcome is independent of r, given our continuity assumption. Then the effect of an outside alternative a is totally captured by this evaluation, because if two problems share the same future outcomes (in case of no agreement now), then the bargaining outcome to day must be the same. Indeed, we postulate that the solution would remain the same, if two problems have the same *end phase evaluation* given S and r. We formulate this as an independent axiom because it has a clear interpretation, although later it turns out that it is implied by other axioms.

End Phase Evaluation Property ($EPEP$). For each $(S, a) \in \Gamma$, the end phase evaluation $g(S, a)$ exists and is independent of r. If $g(S, a) = g(S, a')$, then $\varphi(S, a, T, r) = \varphi(S, a', T, r)$.

Note that combined with the earlier properties, g should be a continuous, efficient, and scale invariant mapping on Ψ.

Next we modify the *independence of irrelevant alternatives* axiom (IIA) to our dynamic setting.

Time Path IIA ($TPIIA$). Let $B = (S, a, T, r) \in \Psi$ and $B' = (S', a', T, r) \in \Psi$ be such that $g(S, a) = g(S', a')$ and $S' \subset S$. If for any $T', 0 < T' \leq T$, $\varphi((S, a, T', r)) \in S'$, then $\varphi(B) = \varphi(B')$.

This property reflects the backward induction principle, and so the justification of our IIA property in the dynamic setting may be more palatable than the the justification of the ordinary IIA in static framework. Note that $TPIIA$ implies $EPEP$.

Next we introduce properties concerning time dimension of the problems.

Time Decomposability (TD). Let $B = (S, a, T, r)$ and $B' = (S, a, T', r)$ with $T > T'$. Then $\varphi(B) = \varphi(S, \varphi(B'), T - T', r)$.

TD states that the solution is decomposable along the time dimension too. The problems B and B' are otherwise the same except that in B there is more time to bargain ($T > T'$). Then players could solve B' first, and use it's solution as an outside alternative in the new problem where the dead line is at $T - T'$.

Next we formulate a symmetry property in our dynamic context. Let $\hat{\pi}$ be the nontrivial permutation on $\{1, 2,$ *i.e.*, $\hat{\pi}(i) = j, i \neq j$. The induced permutation on $\mathbb{R}^2$ is denoted by

π so that $\pi(x_1, x_2) = (x_2, x_1)$ for any $(x_1, x_2) \in \mathbb{R}^2$. An element $x \in \mathbb{R}^2$ is *symmetric* if $\pi(x_1, x_2) = (x_1, x_2)$, and a set $S \subset \mathbb{R}^2$ is symmetric if $\pi(S) = \{\pi(x) \mid x \in S\}$ satisfies $\pi(S) = S$. A subset $S \subset \mathbb{R}^2$ is *symmetric relative to* $a \in \mathbb{R}^2$, if $S - a$ is symmetric.

Dynamic Symmetry (DS). Given $B = (S, a, T, r) \in \Psi$, suppose $S \cap \{x \in \mathbb{R}^2_+ \mid x \geq a\}$ is symmetric relative to $e^{-rt}a$, for all $t \leq T$. Then $\{f(B)\}$ is symmetric relative to $e^{-rt}a$.

This requirement says that if relative symmetry holds all the way from the end phase to the initial phase, then the solution is determined according to the translated symmetry condition. The precondition in DS is rather demanding, but when it is satisfied this axiom becomes very stringent. In fact it can be applied only to problems with linear Pareto frontiers unless a itself is symmetric.

We state our result for any end phase evaluation compatible with properties listed above. But for the sake of simplicity, we write down two more properties for the end phase evaluation g.

Individual Rationality (IR). $g(S, a) \geq a$.

Symmetry (Sym). If $S - a$ is symmetric, then $\{g(S, a)\}$ is symmetric with respect to the point a.

Also, in order to extend the limit solution to problems with non-differentiable Pareto frontiers, we have to extend our definition of the local Nash solution N to this class. This causes no problems and we omit the details.

Theorem 2 *Suppose that g satisfies E, Cont, SI, Sym, and IR. Given g, there is a unique solution, φ^g, satisfying E, Cont., SI, TD, TPIIA, and DS which is given by $\varphi^g(S, a, T, r) = (z_1, F(z_1))$ satisfying $T = \int_{g_1(S,a)}^{z_1} \frac{1}{r\{N_1(z_1, F(z_1)) - z_1\}} dt$ if $g(S, a) \neq N^*(S)$,and $\varphi^g(S, a, T, r) = N^*(S)$ if $g(S, a) = N^*(S)$.*

Proof. It is clear that there are many mappings g satisfying E, $Cont$, SI, Sym and IR. Given such a g, the solution φ^g given above is well defined by Theorem 1. Let us show first that φ^g satisfies the axioms E, $Cont$, SI, TD, $TPIIA$ and DS.

The axioms E and SI are clearly satisfied by φ^g. Let $B = (S, a, T, r) \in \Psi$ be any problem. Note that if a does not satisfy $a_1 = a_2$, then DS is applicable only if S is symmetric with a linear Pareto frontier and φ^g clearly satisfies DS. Continuity of g implies continuity of φ^g because the value of the integral defining φ^g depends continuously on $g_1(S, a)$, z_1 and $N_1(z_1, F(z_1))$. Let $B' = (S', a', T, r)$ be another problem that is related to B as in the statement of $TPIIA$. Since $g(S, a) = g(S', a')$ and the segment of the Pareto frontier connecting $g(S, a)$ to $\varphi^g(B)$ is the same in both problems, $TPIIA$ is satisfied. The axiom TD is satisfied because an integral is an additive function of its integration limits. Therefore φ^g satisfies all the axioms.

Let φ be any solution satisfying axioms E, $Cont$, SI, TD, $TPIIA$ and DS, given that g satisfies axioms E, $Cont$, SI, Sym and IR. We have to show that $\varphi = \varphi^g$.

For a symmetric problem B with a linear Pareto frontier, we have $\varphi(B) = \varphi^g(B)$ by DS and E. By SI, the same result carries to all problems with a linear Pareto frontier. Then consider any two problems $B = (S, a, T, r)$ and $B' = (S', a', T, r)$ with the same *end phase evaluations*, or $g(S, a) = g(S', a')$, and with the same segment of the Pareto frontier joining $g(S, a)$ and $\varphi(B)$. If this segment is a straight line segment, then $TPIIA$ guarantees that $\varphi = \varphi^g$.

Take then any problem $B = (S, a, T, r)$ with the Pareto frontier consisting of two linear pieces L_1 and L_2. If $g(S, a)$ and $\varphi(B)$ are in the same segment L_i, then $\varphi(B) = \varphi^g(B)$. Suppose then that $g(S, a) \in L_1$ and $\varphi(B) \in L_2$. Let $B' = (S, a, T', r)$ and choose T', $0 < T' < T$ in such a way that $\varphi(B') \in L_1 \cap L_2$. That is, the solution $\varphi(B')$ of B' is precisely at the kink of the Pareto frontier of B. This can be done thanks to continuity of φ. Applying TD we get $\varphi(B) = \varphi^g(B)$. By induction, $\varphi(B'') = \varphi^g(B'')$ for any problem B'' with the Pareto frontier consisting of finitely many linear pieces.

Let finally $B = (S, a, T, r) \in \Psi$ be an arbitrary problem. There exists a problem $B' = (S', a, T, r)$ with a linear Pareto frontier supporting the Pareto set of B at $g(S, a) = g(S', a)$. Given a natural number $n > 0$, one can find a problem $B^n = (S^n, a, T, r)$ within a distance $1/n$ (in the Hausdorff metric) from B such that (i) the Pareto frontier of B^n of finitely many linear pieces and (ii) the Pareto frontier of B^n is the same as the Pareto frontier of B' within a sufficiently small neighbourhood of $g(S, a)$. By the previous paragraph, $\varphi(B^n) = \varphi^g(B^n)$. By continuity, $\varphi(B) = \varphi^g(B)$. ■

To single out our solution, we utilize axioms for g on Γ adopted by Salonen (1988).

Covariance (Cov). For any $(S, a) \in \Gamma$ and a positive affine transformation (α, β) of payoffs, if $(S', a') = ((\alpha, \beta)S, (\alpha, \beta)a)$, then $g(S', a') = (\alpha, \beta)g(S, a)$.

Independence of Individually Irrational Alternatives ($IIIA$). For all (S, a), $(S', a) \in \Gamma$, if $S \cap (a + \mathbb{R}^2_+) = S' \cap (a + \mathbb{R}^2_+)$, then $g(S, a) = g(S', a)$.

Given a problem $(S, a) \in \Gamma$, the *ideal point* $M(S, a) \in \mathbb{R}^2_+$ of (S, a) is defined by $M_i(S, d) = \max \{y_i \mid y \in S, y \geq a\}, i = 1, 2$.

Weak Decomposability (WD). Let $(S, a), (S', a) \in \Gamma$ be any two problems such that $S \subset S'$ and $M(S, a) = M(S', a)$. Then there is a problem (S'', a) with $M(S'', a) = M(S, a)$ such that $g(S, a) = g(S, g(S'', a))$ and $g(S', a) = g(S', g(S'', a))$.

This property says that given any two problems with the same ideal points and outside alternatives, one can find a third problem with the same ideal point and outside option in such a way, that the solution can be used as a new outside option for the first two games without changing their solution. One can easily see that WD is satisfied by the (discrete) Raiffa solution R.

Lemma 3 *(Salonen(1988)): There is a unique g which satisfies Sym, Cov, E, IIIA and WD, which coincides with R.*

Corollary 1 *Suppose that g satisfies E, Cont, Cov, Sym, IIIA, and WD. Given g, the unique solution φ satisfying E, Cont., SI, TD, TPIIA, and DS is φ^g, where $g = R$ is the Raiffa solution.*

Apparently, if one wishes to derive the Nash bargaining solution as an *end phase evaluation* g, then requiring g to satisfy IIA (or its alternatives to characterize the Nash bargaining solution; *c.f.* Dagan, Volij, and Winter (2002) and references therein) would yield the desired result. Such a solution would be convenient for application as the solution φ^g would be a sort of "convex" combination of the Nash solution with respect to the best alternative and the one with respect to the origin. As we mentioned earlier, the strategic foundations of such a solution are not yet clear (except for the renowned Nash smoothing argument (*c.f.* Nash (1953) and Van Damme (1987)).

All our results hold for n -person games as well. The main adjustement needed in the proofs is that the integral should be replaced by a line integral in the formula of the limit outcome.

5 Conclusion

In this paper, we investigated the limit solution of the subgame perfect equilibrium of the sequential bargaining game with a deadline. The solution represents the bargaining outcome when players can exchange offers and counteroffers infinitely often within a limited amount of time. The outcome can be represented by a formula implying that under the random proposer protocol with an equal probability, the outcome is close to the Raiffa solution when the deadline is imminent, which we refer to as an end phase evaluation, and the solution tends toward the time preference Nash solution as the deadline is moved further ahead. We also gave an axiomatic foundation for this solution with a strong symmetry reqirement. Although one may obtain the Nash bargaining solution with respect to the best alternative outcome as an end phase evaluation under the *IIA*, its strategic foundation in line with the sequential bargaining game is yet to be found.

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